On the construction of generalized Grassmann representatives of state vectors

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Abstract

Generalized Z_k-graded Grassmann variables are used to label coherent states related to the nilpotent representation of the q-oscillator of Biedenharn and Macfarlane when the deformation parameter is a root of unity. These states are then used to construct generalized Grassmann representatives of state vectors.

Recently in [1] we have constructed coherent states for k-fermions using Kerner's [2] Z_3-graded extension of the Grassmann variables [3]. These results were obtained in the case where the deformation parameter is a primitive cubic root of unity, i.e., $k = 3$. In order to obtain similar results in the generic case (by generic we mean here that q, the deformation parameter, is an arbitrary k^{th} root of unity , i.e., for an arbitrary positive integer k) one should use Z_k-graded generalizations of the Grassmann variables. Unfortunately, up to our knowledge, such structures have not been yet constructed in the spirit of Kerner's variables. There exist however in the literature another point of view and other generalized Z_k-graded Grassmann variables [4].

In this letter we investigate on the use of these latter variables for the description of k-fermions. Namely, we will construct k-fermionic coherent states labeled by Z_k-graded Grassmann variables. The coherent states will be used to derive a space of (Z_k-graded) Grassmann representatives in which state vectors are represented as "holomorphic" functions of the Grassmann variable. There exist many deformations of the harmonic oscillator which, for some values of the deformation parameter, give rise to k-fermions. In this letter we will illustrate the construction using the oscillator deformation of Biedenharn [5] and Macfarlane [6]. This q-oscillator is described by the operators $\{N, a, a^+\}$ with the following relations

$$aa^+ - q^{\pm 1}a^+a = q^{\mp N} , \quad [N, a] = -a , \quad [N, a^+] = a^+ . \tag{1}$$

When q is a kth-root of unity, i.e., $q = \exp\left(\frac{2\pi i}{k}\right)$, this oscillator admits nilpotent representations (see e.g. [7]) in which we have

$$(a)^k = (a^+)^k = 0 . \tag{2}$$

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Note the following useful formulas, valid when q is k^{th}-root of unity,

$$q^k = 1 , \quad \bar{q} = q^{-1} = q^{k-1} , \quad 1 + q + \dots q^{k-1} = 0 .$$ (3)

The Fock space of this representation is finite dimensional of dimension k, with basis

$$\{|n\rangle = \frac{(a^+)^n}{([n]_q)^{\frac{1}{2}}} , \quad a|0\rangle = 0 , \quad n = 0, 1, ..., k\}$$ (4)

on which the different operators act as follows

$$\begin{aligned} a|n\rangle &= ([n]_q)^{\frac{1}{2}} |n-1\rangle \\ a^+|n\rangle &= ([n+1]_q)^{\frac{1}{2}} |n+1\rangle \\ N|n\rangle &= n |n\rangle \end{aligned}$$ (5)

Next we remind the generalized Z_k-graded Grassmann variables [4]. They obey the following q-commutation relations:

$$\begin{aligned} \xi_i \xi_j &= q \xi_j \xi_i \\ (\xi_i)^k &= 0 \end{aligned} \qquad i, j = 1, 2, \dots, d. \quad i < j$$ (6)

d being the dimension of the Grassmann algebra, and q is the same as the deformation parameter of the q-oscillators we are using.

The ξ's are attributed the grade 1 and one introduces $p-1$ grade variables which are hermitian conjugates of the ξ's: $(\xi)^\dagger = \bar{\xi}$, and obey similar relations:

$$\begin{aligned} \bar{\xi}_i \bar{\xi}_j &= q \bar{\xi}_j \bar{\xi}_i \\ (\bar{\xi}_i)^k &= 0 . \end{aligned} \qquad i < j$$ (7)

We have also the following relations between the two sectors

$$\begin{aligned} \xi_i \bar{\xi}_j &= \bar{q} \bar{\xi}_j \xi_i \\ \bar{\xi}_i \xi_j &= \bar{q} \xi_j \bar{\xi}_i . \end{aligned} \qquad i < j$$ (8)

All these relations (6, 7, 8) can be written in a condensed form:

$$\alpha_i \beta_j = q^{ab} \beta_j \alpha_i \quad i < j$$ (9)

where a and b are respectively the grades of α and β; $(\alpha, \beta = \xi, \bar{\xi})$.

In the following we will confine our selves to the one dimensional case, i.e., $d = 1$; which means that we will drop the indices and deal with one ξ, one $\bar{\xi}$ and the one-mode q-oscillator (1, 2).

We will also need the following rules of integration [4], which are generalizations of the Berezin rules for the ordinary Grassmann variable, [3]:

$$\int d\alpha \, \alpha^n = \delta_{n,k-1} .$$ (10)

where $\alpha = \xi, \bar{\xi}$ and n is any positive integer. And we have the following relations

$$\begin{aligned} \xi d\bar{\xi} &= q \, d\bar{\xi}\xi & \bar{\xi} d\xi &= q \, d\xi\bar{\xi} \\ \xi d\xi &= \bar{q} \, d\xi\xi & \bar{\xi} d\bar{\xi} &= \bar{q} \, d\bar{\xi}\bar{\xi} \\ d\xi d\bar{\xi} &= \bar{q} \, d\bar{\xi}d\xi . \end{aligned}$$ (11)

Note that these rules (10, 11) allow to compute the integral of any function over the Grassmann algebra since any such function is written as a finite power series in ξ and $\bar{\xi}$:

$$f(\xi, \bar{\xi}) = \sum_{i,j=0}^{k-1} C_{i,j} \xi^i \bar{\xi}^j . \tag{12}$$

In the fermionic case ($k = 2$) we know that ξ and $\bar{\xi}$ anticommute with the fermionic creation and annihilation operators. Thus, in the generic case (arbitrary k), rather than assuming that the Grassmann variables commute with the k-fermionic operators as is usually done in the literature (see e.g. [8]), one should assume q-commutation relations between ξ, $\bar{\xi}$ and a, a^+:

$$
\begin{array}{llll}
\xi a^+ & = & q\, a^+ \xi \qquad\qquad & \xi a & = & \bar{q}\, a\xi \\
\bar{\xi} a^+ & = & \bar{q}\, a^+ \bar{\xi} & \bar{\xi} a & = & q\, a\bar{\xi} .
\end{array}
\tag{13}
$$

In analogy with the fermionic case and the Z_3-graded case considered in [1], where coherent states are constructed by acting on the vacuum $|0\rangle$ with the exponential function (in the former case) and q-exponential (in the latter), coherent states for k-fermions are obtained similarly:

$$|\xi\rangle_k = \sum_{n=0}^{k-1} \frac{(a^+\xi)^n}{[n]_q!} |0\rangle := \exp_q(a^+\xi) |0\rangle . \tag{14}$$

Using (13) one can find the following relation

$$(a^+\xi)^n = \bar{q}^{\frac{n(n+1)}{2}} \xi^n (a^+)^n \tag{15}$$

which allows us to expand the coherent states (14) in the basis (4) with Grassmann coefficients:

$$|\xi\rangle_k = \sum_{n=0}^{k-1} \bar{q}^{\frac{n(n+1)}{2}} \frac{\xi^n}{\left([n]_q!\right)^{\frac{1}{2}}} |n\rangle . \tag{16}$$

Next we will prove that, as any harmonic oscillator coherent state, these states are eigenstates of the annihilation operator with ξ as eigenvalue:

$$
\begin{aligned}
a|\xi\rangle_k & = \sum_{n=0}^{k-1} \bar{q}^{\frac{n(n+1)}{2}} \frac{a\xi^n}{\left([n]_q!\right)^{\frac{1}{2}}} |n\rangle \\
& = \sum_{n=0}^{k-1} \bar{q}^{\frac{n(n-1)}{2}} \frac{\xi^n a}{\left([n]_q!\right)^{\frac{1}{2}}} |n\rangle \\
& = \sum_{n=1}^{k-1} \bar{q}^{\frac{n(n-1)}{2}} \frac{\xi^n}{\left([n-1]_q!\right)^{\frac{1}{2}}} |n-1\rangle ,
\end{aligned}
\tag{17}
$$

where we have used (13) to obtain the second equality and $\{a|n\rangle = [n]_{\bar{q}}^{\frac{1}{2}}|n-1\rangle,\ a|0\rangle = 0\}$ for the third one. Then making the change $\{m = n - 1\}$ and the nilpotency of ξ: $\xi^k = 0$, we obtain the desired result:

$$
\begin{aligned}
a|\xi\rangle_k & = \sum_{m=0}^{k-1} \bar{q}^{\frac{m(m+1)}{2}} \frac{\xi^{m+1}}{\left([m]_q!\right)^{\frac{1}{2}}} |m\rangle \\
& = \xi|\xi\rangle_k .
\end{aligned}
\tag{18}
$$

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Coherent states are usually not mutually orthogonal, this is also the case for the coherent states (16); let us compute the scalar product of two such states:

$$
\begin{aligned}
{}_k\langle\xi_2|\xi_1\rangle_k &= \sum_{n,m=0}^{k-1} \bar{q}^{-\frac{n(n+1)}{2}} q^{-\frac{m(m+1)}{2}} \frac{1}{\left([n]_q!\right)^{\frac{1}{2}}\left([m]_q!\right)^{\frac{1}{2}}} \langle m|\bar{\xi}_2^m \xi_1^n|n\rangle \\
&= \sum_{n=0}^{k-1} \frac{\bar{\xi}_2^n \xi_1^n}{[n]_q!}
\end{aligned}
\tag{19}
$$

where we have used the orthonormality of the basis (4): $\langle m|n\rangle = \delta_{m,n}$. We make use of the equality

$$
(\bar{\xi}\xi)^n = \bar{q}^{-\frac{n(n-1)}{2}} \bar{\xi}^n \xi^n
\tag{20}
$$

to write the scalar product in the form

$$
{}_k\langle\xi_2|\xi_1\rangle_k = \sum_{n=0}^{k-1} q^{\frac{n(n-1)}{2}} \frac{(\bar{\xi}_2\xi_1)^n}{[n]_q!} \ .
\tag{21}
$$

Then using the following relations between the two box functions $[n]_q$ and $\{n\}_Q = \frac{1-Q^n}{1-Q}$ see e.g. [9] (note that $\{n\}_{\bar{q}} \neq \{n\}_q$ whereas $[n]_{\bar{q}} = [n]_q$):

$$
[n]_q = [n]_{\bar{q}} = \bar{q}^{n-1}\{n\}_{q^2} = q^{n-1}\{n\}_{\bar{q}^2} \ ,
\tag{22}
$$

the corresponding q-factorials are then related by the following:

$$
[n]_q! = q^{\frac{n(n-1)}{2}}\{n\}_{\bar{q}^2}! \ .
\tag{23}
$$

This permits us to rewrite the scalar product (21) as follows:

$$
\begin{aligned}
{}_k\langle\xi_2|\xi_1\rangle_k &= \sum_{n=0}^{k-1} \frac{(\bar{\xi}_2\xi_1)^n}{\{n\}_{\bar{q}^2}!} \\
&:= E_{\bar{q}^2}(\bar{\xi}_2\xi_1) \ ,
\end{aligned}
\tag{24}
$$

where we have used the following q-exponential [9]

$$
E_q(\alpha) = \sum_{n=0}^{k} \frac{\alpha^n}{\{n\}_q!} \ ; \quad \{n\}_q! = \{1\}_q \ldots \{n\}_q \ ; \quad \{0\}_q! := 1 \ .
\tag{25}
$$

One of the most important defining properties of coherent states is that they provide a resolution of unity [10]. Thus in order to prove that the states (16) are indeed coherent states we shall show that they allow a resolution of unity. We shall look for such a resolution in the form

$$
\iint d\bar{\xi}d\xi \ \omega(\bar{\xi}\xi) \ |\xi\rangle_{k\,k}\langle\xi| = I \ ,
\tag{26}
$$

where the weight function is written as follows

$$
\omega(\bar{\xi}\xi) = \sum_{n=0}^{k-1} c_n \ \xi^n\bar{\xi}^n \ .
\tag{27}
$$

We must compute the coefficients c_n such that the equality in (26) holds.

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Using (16) and (27) the left hand side of (26) is written as follows

$$\iint \sum_{l,n,p=0}^{k-1} d\bar{\xi}d\xi \, c_n \xi^n \bar{\xi}^n \, \bar{q}^{\frac{l(l+1)}{2}} q^{\frac{p(p+1)}{2}} \frac{\xi^l \bar{\xi}^p}{\left([l]_q!\right)^{\frac{1}{2}} \left([p]_q!\right)^{\frac{1}{2}}} \, |l\rangle\langle p| \, . \tag{28}$$

Then taking account of relations (8) and the integration rules (10), it becomes

$$\iint d\bar{\xi}d\xi \sum_{l,n=0}^{k-1} \frac{c_n q^{nl}}{[l]_q!} \, \xi^{n+l}\bar{\xi}^{n+l}|l\rangle\langle l| \, . \tag{29}$$

The Fock space basis being complete: $\sum_{l=1}^{k-1}|l\rangle\langle l| = I$, now using (10) we obtain the following constraints on the coefficients c_n

$$I = \sum_{\substack{n,l=0 \\ n+l=k-1}}^{k-1} \frac{c_n q^{nl}}{[l]_q!} \iint d\bar{\xi}d\xi \xi^{n+l}\bar{\xi}^{n+l} \, |l\rangle\langle l| \, , \tag{30}$$

i.e.,

$$\frac{c_n q^{nl}}{[l]_q!} = 1 \quad \text{and} \quad n+l = k-1 \, . \tag{31}$$

We have thus found the coefficients c_n, for which the equality (26) holds, to be

$$\begin{aligned} c_n &= \bar{q}^{n(k-n-1)}[k-n-1]! \\ &= q^{n(n+1)}[k-n-1]! \end{aligned} \tag{32}$$

The weight function appearing in the resolution of unity (26) is therefore given by

$$\begin{aligned} \omega(\bar{\xi}\xi) &= \sum_{n=0}^{k-1} c_n \xi^n \bar{\xi}^n \\ &= \sum_{n=0}^{k-1} q^{n(n+1)}[k-n-1]_q! \xi^n \bar{\xi}^n \\ &= \sum_{n=0}^{k-1} q^{\frac{n(n+1)}{2}}[k-n-1]_q!(\bar{\xi}\xi)^n \, . \end{aligned} \tag{33}$$

Coherent states allow in general the mapping of vectors of the underlying Hilbert space into holomorphic functions (Bargmann-Fock space), when they (the coherent states) are parameterized by a complex variable. In the case where they are parameterized by Grassmann variables (fermionic case [10, 11] or k-fermionic case [1]), coherent states allow the mapping into a space of Grassmann representatives. In all cases it is the resolution of unity that permits this mapping.

However, when dealing with generalized Grassmann variables, it was shown in [1] that it is more convenient, and sometimes essential, to use another form of the resolution of unity, namely

$$\iint |\xi\rangle_k \, d\bar{\xi}d\xi \, \tilde{\omega}(\bar{\xi}\xi) \, {}_k\langle\xi| = I \, , \tag{34}$$

where the weight function $\tilde{\omega}$ is written in the same form as ω in (27), with *a priori* different coefficients:

$$\tilde{\omega}(\bar{\xi}\xi) = \sum_{n=0}^{k-1} \tilde{c}_n \xi^n \bar{\xi}^n \, . \tag{35}$$

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We have deliberately given here the two forms (26, 34)of the resolution of unity to insist on the fact that in the case we are considering in this letter these two forms are equivalent, whereas in [1] (where we used the Z_3-graded Grassmann variables of Kerner [2]) they are not. By equivalent we mean that one can deduce one form from the other. So, the coefficients $\tilde{c}_n$ in (35) can be evaluated either by deducing them from the coefficients c_n directly or by explicitly performing the calculations as for the c_n. In any case the result is the following:

$$
\begin{aligned}
\tilde{c}_n &= [k-n-1]_q! \\
&= \bar{q}^{n(n+1)} c_n .
\end{aligned}
\tag{36}
$$

These are the values for which the equality in (34) holds.

We are now in a position to construct the Grassmann representatives of state vectors. Indeed, using the latter form of the resolution of unity (34), any vector $|\psi\rangle$ in the Hilbert space spanned by the basis $\{|n\rangle,\ n = 0, 1, \ldots, k-1\}$ can be determined by its Grassmann representative $\psi(\bar{\xi}) := {}_k\langle\xi|\psi\rangle$:

$$
|\psi\rangle = \iint |\xi\rangle_k d\bar{\xi} d\xi \ \tilde{\omega}(\bar{\xi}\xi) \ \psi(\bar{\xi}) .
\tag{37}
$$

In particular, the basis vectors are realized by the following monomials in $\bar{\xi}$:

$$
n(\bar{\xi}) := \langle\xi|n\rangle = \bar{q}^{\frac{n(n-1)}{2}} \frac{\bar{\xi}^n}{\left([n]_q!\right)^{\frac{1}{2}}} ,
\tag{38}
$$

which, therefore, constitute a basis of the space of *polynomial* functions over the Grassmann algebra generated by $\bar{\xi}$ also called space of (generalized) Grassmann representatives.

The scalar product in this realization space is obtained using (34) and is given by

$$
\langle\psi|\varphi\rangle = \iint \bar{\psi}(\xi) \ d\bar{\xi} d\xi \ \tilde{\omega}(\bar{\xi}\xi) \ \varphi(\bar{\xi}) .
\tag{39}
$$

We check that, in particular, the orthonormality of the basis $\{|n\rangle,\ n = 0, 1, \ldots, k-1\}$ is preserved under this scalar product:

$$
\begin{aligned}
\langle m|n\rangle &= \iint \bar{m}(\xi) \ d\bar{\xi} d\xi \ \tilde{\omega}(\bar{\xi}\xi) \ n(\bar{\xi}) \\
&= \iint \sum_{l=0}^{k-1} \frac{q^{\frac{m(m-1)}{2}}}{\left([m]_q!\right)^{\frac{1}{2}}} \frac{\bar{q}^{\frac{n(n-1)}{2}}}{\left([n]_q!\right)^{\frac{1}{2}}} \ \tilde{c}_l \xi^m \ d\bar{\xi} d\xi \ \xi^l \bar{\xi}^l \bar{\xi}^n \\
&= \iint \sum_{i=0}^{k-1} \frac{\tilde{c}_l}{[n]_q!} \delta_{m,n} \ d\bar{\xi} d\xi \ \xi^{l+n} \bar{\xi}^{l+n} \\
&= \sum_{i=0}^{k-1} \frac{\tilde{c}_l}{[n]_q!} \delta_{m,n} \ \delta_{l,k-n-1} \\
&= \frac{\tilde{c}_{k-n-1}}{[n]_q!} \delta_{m,n} \\
&= \delta_{m,n} .
\end{aligned}
\tag{40}
$$

In the last equality, we have used the fact that

$$
\tilde{c}_{k-n-1} = [n]_q! .
\tag{41}
$$

Moreover, if we denote by $\partial_{\bar{\xi}}$ the (ordinary) derivative operator:

$$
\partial_{\bar{\xi}} \ \bar{\xi}^m = m\bar{\xi}^{m-1}
\tag{42}
$$

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and $D_{\bar{\xi}}$ the deformed derivative operator:

$$D_{\bar{\xi}} \, \bar{\xi}^m = [m]_q \, \bar{\xi}^{m-1} \; ; \tag{43}$$

actually this last operator is defined as follows

$$D_{\bar{\xi}} \, f(\bar{\xi}) = \frac{f(q\bar{\xi}) - f(q^{-1}\bar{\xi})}{(q - q^{-1})\bar{\xi}} \; ; \tag{44}$$

then one can easily realize the annihilation and creation operator as differential operators, acting on the space of Grassmann representatives, as follows

$$a \quad \longrightarrow \quad q^{\bar{\xi}\partial_{\bar{\xi}}} \, D_{\bar{\xi}} \tag{45}$$

$$a^+ \quad \longrightarrow \quad \bar{\xi} \, q^{\bar{\xi}\partial_{\bar{\xi}}} \tag{46}$$

As a matter of facts the differential operator $\bar{\xi}\partial_{\bar{\xi}}$ appearing in these expressions is the realization of the number operator N.

It is obvious that in order to obtain a realization in terms of ξ, rather than $\bar{\xi}$, one should use coherent states labeled by $\bar{\xi}$ (rather than ξ), i.e., $|\bar{\xi}\rangle$.

In summary, we have constructed coherent states, which are labeled by Z_k-graded Grassmann variables, relative to the nilpotent representation of the q-oscillator of Biedenharn and Macfarlane when q (the parameter of deformation) is a kth root of unity. We have then showed that these coherent states map the Hilbert space of the representation of the q-oscillator into a space of some *polynomial* functions over the Grassmann variables, i.e., the space of Grassmann representatives. The different operators of the q-oscillator algebra act on this space as differential operators involving both ordinary and deformed derivatives.

At the same footstep, the coherent states constructed in this space can be used to obtain Grassmann representatives of any operator acting on the Hilbert space, using some sort of *Grassmann covariant symbol* of the operator. This should prove useful for the construction of a path integral formalism with Z_k-graded Grassmann variables.

It is worth to mention that our starting point here for the construction of k-fermionic coherent states was the assumption of the non-commutativity of the Grassmann variables with the k-fermionic operators (13), instead of assuming commutativity as is generally the case in the literature [8]. We would like to argue here that this last point of view (i.e., commutativity instead of (13)) does not seem to be consistent. In fact, one of the features of k-fermionic oscillators is that they reduce to the usual fermionic one, when the deformation parameter takes a special value. One expects this property to be preserved when constructing further k-fermionic structures, such as coherent states. Moreover, taking into account that in the usual fermionic case the Grassmann variables do not commute with the fermionic operators but rather anticommute, it is easy to check that this behaviour is recovered in the fermionic limit by adopting the point of view presented in this letter (i.e., $k = 2$) from (13)) but not by assuming commutativity. Therefore, it is easy to convince oneself that the point of view generally adopted in the literature, even though rendering the calculations simpler, is not consistent since it fails to encompass the fermionic behaviour, which we are supposed to generalize by considering k-fermionic structures.

Finally, let us note that we have used the q-oscillator of Biedenharn and Macfarlane in this letter, the construction is however easily extended to other q-oscillators giving rise to k-fermions.

Acknowledgments

References

[1] M. El Baz, Y. Hassouni and F. Madouri; *Z_3-graded Grassmann variables, parafermions and their coherent states*, Phys. Lett. **B536** (2002) 321, math-ph/0206017

[2] R. Kerner; *Z_3-graded algebras and the cubic root of the supersymmetry translations*, J. Math. Phys. **33** (1992) 403

R. Kerner; Z_3-graded exterior differential calculus and gauge theories of higher order, Lett. Math. Phys. **36** (1996) 441, math-ph/0004032

V. Abramov, R. Kerner and B. Le Roy; *Hypersymmetry: a Z_3-graded generalization of supersymmetry*, J. Math. Phys. **38** (1997) 1650, hep-th/9607143

[3] F.A. Berezin; *The method of second quantization*, (Academic Press Inc., N.Y., 1966)

[4] S. Majid; *Random walk and the heat equation on superspace and anyspace*, J. Math. Phys. **35** (1994) 3753

[5] L. C. Biedenharn; *The quantum group $SU_q(2)$ and a q-analogue of the boson operators*, J. Phys. **A22** (1989) L873

[6] A. J. Macfarlane; *On q-Analogues of the quantum harmonic oscillator and the quantum group $SU_q(2)$*, J. Phys. **A22** (1989) 4581

[7] L. C. Biedenharn and M. A. Lohe; *Quantum Group Symmetry and q-Tensor Algebras*, (World Scientific, Singapore, 1995)

S. Majid; *Foundations of quantum group theory*, (Cambridge Univ. Press, 1995)

[8] M. Daoud and M. Kibler; *A fractional supersymmetric oscillator and its coherent states*, preprint: math-ph/9912024

[9] M. Arik and D. D. Coon; *Hilbert spaces of analytic functions and generalized coherent states*, J. Math. Phys. **17** (1976) 524

[10] J.R. Klauder and B-S. Skagertan; *Coherent States*, (World Scientifique, Singapore, 1985)

A.M. Perelomov; *Generalized Coherent States and Their Applications*, (Springer-Verlag, Berlin 1986)

S.T. Ali, J-P. Antoine and J-P. Gazeau; *Coherent States, Wavelets and Their Generalizations*, (Springer-Verlag, New York, 2000)

[11] Y. Ohnuki and T. Kashiwa; *Coherent states of Fermi operators and the path integral*, Prog. Theor. Phys. **60** (1978) 548